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Large deviation principles for 2-D stochastic Navier–Stokes equations driven by Lévy processes

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Abstract

In this paper, we establish a large deviation principle for the two-dimensional stochastic Navier–Stokes equations driven by Lévy processes, which involves the study of the Lévy noise and the investigation of the effect of the highly nonlinear, unbounded drifts.

Keywords: Stochastic Navier–Stokes equation; Lévy process; Large deviation principle

1. Introduction

It is well known that the two-dimensional Navier–Stokes equation with Dirichlet boundary condition describes the time evolution of an incompressible fluid and is given by

$$\begin{cases} du - \nu\Delta u\,dt + (u\cdot\nabla)u\,dt + \nabla p\,dt = g\,dt, \\ (\nabla\cdot u)(t,x) = 0, & \text{for } x\in D,\ t>0, \\ u(t,x) = 0, & \text{for } x\in\partial D,\ t>0, \\ u(0,x) = u_0(x), & \text{for } x\in D, \end{cases}$$

where D is a bounded domain in $\mathbb{R}^2$ with smooth boundary ∂D, $u(t,x)\in\mathbb{R}^2$ denotes the velocity field at time t and position x, $p(t,x)$ denotes the pressure field, $\nu>0$ is the viscosity and g is a deterministic force.

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To formulate the Navier–Stokes equations, we introduce the following standard spaces:

$$V = \{v \in H_0^1(D; \mathbb{R}^2):\ \nabla \cdot v = 0,\ \text{a.e. in } D\},$$

with the norm

$$\|v\|_V := \left(\int_D |\nabla v|^2\,dx\right)^{\frac{1}{2}} = \|v\|,$$

and denote by $(, . ,)$ the inner product of V. H is the closure of V in the L^2-norm

$$|v|_H := \left(\int_D |v|^2\,dx\right)^{\frac{1}{2}} = |v|.$$

The inner product on H will be denoted by $(\cdot\,,\cdot)$.

Define the operator A (Stokes operator) in H by the formula

$$Au := -\nu P_H \Delta u, \quad \forall u \in H^2(D; \mathbb{R}^2) \cap V,$$

where the linear operator P_H (Helmhotz–Hodge projection) is the projection operator from $L^2(D; \mathbb{R}^2)$ to H, and the nonlinear operator B

$$B(u, v) := P_H\big((u \cdot \nabla)v\big),$$

with the notation $B(u) = B(u, u)$. Obviously the domain of B requires that $(u \cdot \nabla)v$ belongs to the space $L^2(D; \mathbb{R}^2)$.

By applying the operator P_H to each term of the above Navier–Stokes equation (NSE), we can rewrite the NSE in the following abstract form:

$$du(t) + Au(t)\,dt + B\big(u(t)\big)\,dt = f(t)\,dt \quad \text{in } L^2\big([0, T]; V'\big), \tag{1.1}$$

with the initial condition

$$u(0) = x \quad \text{in } H. \tag{1.2}$$

The purpose of this paper is to establish a large deviation principle for Eq. (1.1) driven by the additive Lévy noise, that is

$$\begin{cases} du^n(t) = -Au^n(t)\,dt - B\big(u^n(t)\big)\,dt + \mathrm{b}\,dt + \dfrac{1}{\sqrt{n}}\,dW(t) + \dfrac{1}{n}\displaystyle\int_X f(x)\tilde{N}_n(dt, dx), \\ u^n(0) = x \in H, \end{cases} \tag{1.3}$$

where $W(\cdot)$ is an H-valued Brownian motion, b is a constant vector in H, f is a measurable mapping from some measurable space X to H, and $\tilde{N}_n(dt, dx)$ is a compensated Poisson measure on $[0, \infty) \times X$ with intensity measure $n\nu$, where ν is a σ-finite measure on $\mathcal{B}(X)$.

There exists a great amount of literature on the stochastic Navier–Stokes equation. A good reference for stochastic Navier–Stokes equations driven by additive noise is the book [5] and the references therein. The existence and uniqueness of solutions for the 2-D stochastic Navier–Stokes equations with multiplicative Gaussian noise were obtained in [11,15,17]. The ergodic properties and invariant measures of the 2-D stochastic Navier–Stokes equations were studied in [10] and [13]. The small Gaussian noise large deviation of the 2-D stochastic Navier–Stokes equations was established in [17] and the large deviation of occupation measures was considered in [12].

Large deviations for stochastic equations and stochastic partial differential equations have been investigated in many papers, see [1–4,19]. There is not much work on large deviations for stochastic evolution equations driven by Lévy noises in infinite dimensions. To the best of our knowledge, [16] is the first paper on this topic, where the Lipschitz coefficients are considered. For this paper, in addition to the difficulties caused by the Lévy noise, much of the problem is to deal with the highly nonlinear term $B(u,u)$. For this purpose, we need to prove a number of exponential estimates for the energy of the solutions as well as the exponential convergence of the approximating solutions. We mention that the large deviation principle for the solution of the stochastic equation driven by jump processes in finite dimensions has been established in [8].

The organization of this paper is as follows. In Section 2, we collect some preliminary facts which are frequently used in the sequel. In Section 3, we prove a number of exponential estimates for the solutions, which will play an important role in the rest of the paper. Section 4 is devoted to establish a large deviation principle.

2. Preliminaries

Identifying H with its dual H', we consider Eq. (1.3) in the framework of Gelfrand triple:

$$V \subset H \cong H' \subset V'.$$

In this way, we may consider A as a bounded operator from V into V'. Moreover, we also denote by $\langle \cdot,\cdot\rangle$, the duality between V and V'. Hence, for $u=(u_i)\in V$, $w=(w_i)\in V$, we have

$$\langle Au, w\rangle = \nu \sum_{i,j} \int_D \partial_i u_j \partial_i w_j \, dx = \nu((u,w)). \tag{2.1}$$

Introduce a trilinear form on $H\times H\times H$ by setting

$$b(u,v,w) = \sum_{i,j}^{2} \int_D u_i \partial_i v_j w_j \, dx, \tag{2.2}$$

whenever the integral in (2.2) makes sense. In particular, if $u,v,w\in V$, then

$$\big\langle B(u,v), w\big\rangle = \big\langle (u\cdot\nabla)v, w\big\rangle = \sum_{i,j}^{2} \int_D u_i \partial_i v_j w_j \, dx = b(u,v,w).$$

By integration by parts,

$$b(u,v,w) = -b(u,w,v), \tag{2.3}$$

therefore

$$b(u,v,v) = 0, \quad \forall u, v \in V. \tag{2.4}$$

There are some well-known estimates for b (see [18] for example), which will be required in the rest of this paper and we list them here. Throughout the paper, we denote various generic positive constants by the same letter c. We have

$$\bigl|b(u,v,w)\bigr| \leqslant c\|u\| \cdot \|v\| \cdot \|w\| \tag{2.5}$$

$$\bigl|b(u,v,w)\bigr| \leqslant c|u| \cdot \|v\| \cdot |Aw| \tag{2.6}$$

$$\bigl|b(u,v,w)\bigr| \leqslant c\|u\| \cdot |v| \cdot |Aw| \tag{2.7}$$

$$\bigl|b(u,v,w)\bigr| \leqslant 2\|u\|^{\frac{1}{2}} \cdot |u|^{\frac{1}{2}} \cdot \|w\|^{\frac{1}{2}} \cdot |w|^{\frac{1}{2}} \cdot \|v\| \tag{2.8}$$

for suitable u, v, w. Moreover, combining (2.3) and (2.8), we obtain a useful estimate as follows:

$$\bigl|B(u)\bigr|_{V'} = \sup_{\|v\|\leqslant 1} \bigl|b(u,u,v)\bigr| = \sup_{\|v\|\leqslant 1} \bigl|b(u,v,u)\bigr| \leqslant 2\|u\| \cdot |u|. \tag{2.9}$$

Before ending this section, let us set up the stochastic basis. Let $(\Omega, \mathcal{F}, P)$ be a probability space equipped with a filtration $\{\mathcal{F}_t, t \geqslant 0\}$ satisfying the usual conditions. Let $W(\cdot)$ be a H-valued Brownian motion on $(\Omega, \mathcal{F}, P)$ with the covariance operator Q, which is a positive, symmetric, trace class operator on H. Let $(X, \mathcal{B}(X))$ be a measurable space and $\nu(dx)$ a σ-finite measure on it. Let $p = (p(t), t \in D_p)$ be a stationary $\mathcal{F}_t$-Poisson point process on X with characteristic measure $\nu(dx)$, where D_p is a countable subset of $[0, \infty)$ depending on random parameter ω (see[14]). Denote by $N(dt, dx)$ the Poisson counting measure associated with p, i.e., $N(t, A) = \sum_{s \in D_p, s \leqslant t} I_A(p(s))$. Let $\tilde{N}(dt, dx) := N(dt, dx) - dt\nu(dx)$ be the compensated Poisson measure. Denoted by $\tilde{N}_n(dt, dx)$ the compensated Poisson measure with the characteristic measure $n\nu$. Let b be a constant vector in H and f be a measurable mapping from X to H.

Throughout this paper, we assume that

$$\int_X \bigl|f(x)\bigr|^2 \exp\bigl(a\bigl|f(x)\bigr|\bigr)\nu(dx) < +\infty, \quad \text{for all } a > 0. \tag{2.10}$$

Using approaches similar to that in [17], we can easily show in this additive case that Eq. (1.3) has a unique solution in $L^2([0,T]; V) \cap D([0,T]; H)$, where $D([0,T]; H)$ denotes the space of all the càdàg pathes from $[0,T]$ to H endowed with the uniform convergence topology.

3. Exponential estimates

To establish the large deviation principle, we first prove some exponential estimates.
Let $u^n_\cdot$ be the solution of the following stochastic Navier–Stokes equation

$$u^n_t = x - \int_0^t Au^n_s\,ds - \int_0^t B\big(u^n_s\big)\,ds + \mathrm{b}t + \frac{1}{\sqrt{n}}W_t + \frac{1}{n}\int_0^t\int_X f(x)\tilde{N}_n(ds,dx). \tag{3.1}$$

Let $X^n_\cdot = nu^n_\cdot$, then $X^n_\cdot$ is the solution of the following equation

$$X^n_t = nx - \int_0^t AX^n_s\,ds - \frac{1}{n}\int_0^t B\big(X^n_s\big)\,ds + n\mathrm{b}t + \sqrt{n}W_t + \int_0^t\int_X f(x)\tilde{N}_n(ds,dx). \tag{3.2}$$

Denote by $\{e_k\}_{k=1}^\infty$ an orthonormal basis of H that consists of eigenvectors of Q in V with $\{\lambda_k\}_{k=1}^\infty$ being the corresponding eigenvalues.

Lemma 3.1. *For $g\in C_b^2(H)$, $M^g_t = \exp(g(X^n_t) - g(nx) - \int_0^t h(X^n_s)\,ds)$ is a $\mathcal{F}_t$-local martingale, where*

$$\begin{aligned} h(y) = {} & -\left\langle Ay + \frac{1}{n}B(y), g'(y)\right\rangle + n\big(\mathrm{b}, g'(y)\big) + \frac{n}{2}\sum_{k=1}^\infty \lambda_k\big(\big[g'(y)\otimes g'(y) + g''(y)\big]e_k, e_k\big) \\ & + n\int_X \big\{\exp\big[g\big(y+f(x)\big) - g(y)\big] - 1 - \big(g'(y), f(x)\big)\big\}\nu(dx). \end{aligned} \tag{3.3}$$

Proof. Applying Itô's formula to $\exp(g(X^n_t))$, we get

$$\exp\big(g\big(X^n_t\big) - g(nx)\big) - \int_0^t \exp\big(g\big(X^n_s\big) - g(nx)\big)h\big(X^n_s\big)\,ds$$

is a local martingale. The lemma follows by another integration by parts. □

In the rest of this section, we always set $g(y) := (1+\lambda|y|^2)^{\frac{1}{2}}$ $(\lambda > 0)$. It is easy to see that

$$\sup_y\big|g''(y)\big| \leqslant \lambda, \qquad \sup_y\big|g'(y)\big| \leqslant \lambda^{\frac{1}{2}}.$$

Denote by $Tr\,Q$ the trace of the operator Q, i.e., $Tr\,Q := \sum_{i=1}^\infty (Qe_i, e_i) = \sum_{i=1}^\infty \lambda_i$.
We have the following results.

Lemma 3.2. $\lim_{r\to\infty}\limsup_{n\to\infty}\frac{1}{n}\log P(\sup_{0\leqslant t\leqslant 1}|u^n_t| > r) = -\infty$.

Proof. By the proof of Proposition 4.2 in [16], we know that

$$\left|\int_X \left\{\exp\left[g\left(y+f(x)\right)-g(y)\right]-1-\left(g'(y), f(x)\right)\right\}\nu(dx)\right|$$
$$\leqslant \int_X \lambda\left|f(x)\right|^2 \exp\left(\lambda^{\frac{1}{2}}\left|f(x)\right|\right)\nu(dx) := M_\lambda.$$

Since

$$\left\langle B(y), g'(y)\right\rangle = \frac{1}{2}\frac{2\lambda}{(1+\lambda|y|^2)^{\frac{1}{2}}}\left\langle B(y), y\right\rangle = 0,$$

we have,

$$h\left(X_t^n\right) \leqslant n|\mathrm{b}|\lambda^{\frac{1}{2}} + n\lambda\, Tr\, Q + nM_\lambda, \tag{3.4}$$

where $h(\cdot)$ is defined in Lemma 3.1.

Observe that

$$\begin{aligned}
P\Big(\sup_{0\leqslant t\leqslant 1}\left|u_t^n\right| > r\Big) &= P\Big(\sup_{0\leqslant t\leqslant 1}\left|X_t^n\right| > nr\Big) \\
&= P\Big(\sup_{0\leqslant t\leqslant 1} g\left(X_t^n\right) > \left(1+\lambda n^2 r^2\right)^{\frac{1}{2}}\Big) \\
&= P\left(\sup_{0\leqslant t\leqslant 1}\left[g\left(X_t^n\right) - g(nx) - \int_0^t h\left(X_s^n\right)ds + g(nx) + \int_0^t h\left(X_s^n\right)ds\right]\right. \\
&\qquad \left. > \left(1+\lambda n^2 r^2\right)^{\frac{1}{2}}\right) \\
&\leqslant P\left(\sup_{0\leqslant t\leqslant 1}\left[g\left(X_t^n\right) - g(nx) - \int_0^t h\left(X_s^n\right)ds\right] + g(nx) + \sup_{0\leqslant t\leqslant 1}\int_0^t h\left(X_s^n\right)ds\right. \\
&\qquad \left. > \left(1+\lambda n^2 r^2\right)^{\frac{1}{2}}\right) \\
&\leqslant P\left(\sup_{0\leqslant t\leqslant 1}\left[g\left(X_t^n\right) - g(nx) - \int_0^t h\left(X_s^n\right)ds\right]\right. \\
&\qquad \left. > \left(1+\lambda n^2 r^2\right)^{\frac{1}{2}} - g(nx) - \sup_{0\leqslant t\leqslant 1}\int_0^t h\left(X_s^n\right)ds\right).
\end{aligned} \tag{3.5}$$

Due to (3.4) and Doob's inequality,

$$
\begin{aligned}
&P\left(\sup_{0\leqslant t\leqslant 1}\left(g\left(X_t^n\right)-g(nx)-\int_0^t h\left(X_s^n\right)ds\right)>\left(1+\lambda n^2r^2\right)^{\frac{1}{2}}-g(nx)-\sup_{0\leqslant t\leqslant 1}\int_0^t h\left(X_s^n\right)ds\right)\\
&\quad\leqslant P\left(\sup_{0\leqslant t\leqslant 1}\left[g\left(X_t^n\right)-g(nx)-\int_0^t h\left(X_s^n\right)ds\right]\right.\\
&\qquad\left.>\left(1+\lambda n^2r^2\right)^{\frac{1}{2}}-g(nx)-n|\mathrm{b}|\lambda^{\frac{1}{2}}-n\lambda\, Tr\, Q-nM_\lambda\right)\\
&\quad\leqslant\left(\sup_{0\leqslant t\leqslant 1}E\left[\exp\left(g\left(X_t^n\right)-g(nx)-\int_0^t h\left(X_s^n\right)ds\right)\right]\right)\\
&\qquad\times\exp\left[-\left(1+\lambda n^2r^2\right)^{\frac{1}{2}}+g(nx)+n|\mathrm{b}|\lambda^{\frac{1}{2}}+n\lambda\, Tr\, Q+nM_\lambda\right]\\
&\quad\leqslant\exp\left[-\left(1+\lambda n^2r^2\right)^{\frac{1}{2}}+g(nx)+n|\mathrm{b}|\lambda^{\frac{1}{2}}+n\lambda\, Tr\, Q+nM_\lambda\right],
\end{aligned}
\tag{3.6}
$$

where in the last step, we used the fact that

$$
\sup_{0\leqslant t\leqslant 1}E\left[\exp\left(g\left(X_t^n\right)-g(nx)-\int_0^t h\left(X_s^n\right)ds\right)\right]\leqslant 1,
$$

because $\exp(g(X_t^n)-g(nx)-\int_0^t h(X_s^n)\,ds)$ is a nonnegative local martingale with the initial value 1. Putting (3.5) and (3.6) together, we have

$$
\frac{1}{n}\log P\left(\sup_{0\leqslant t\leqslant 1}\left|u_t^n\right|>r\right)\leqslant-\frac{(1+\lambda n^2r^2)^{\frac{1}{2}}}{n}+\frac{(1+\lambda n^2|x|^2)^{\frac{1}{2}}}{n}+|\mathrm{b}|\lambda^{\frac{1}{2}}+\lambda\, Tr\, Q+M_\lambda. \tag{3.7}
$$

Hence,

$$
\limsup_{n\to\infty}\frac{1}{n}\log P\left(\sup_{0\leqslant t\leqslant 1}\left|u_t^n\right|>r\right)\leqslant-\left(\lambda r^2\right)^{\frac{1}{2}}+\left(\lambda|x|^2\right)^{\frac{1}{2}}+|\mathrm{b}|\lambda^{\frac{1}{2}}+\lambda\, Tr\, Q+M_\lambda.
$$

Taking $r\to\infty$ in the above inequality, one obtains the result. □

Lemma 3.3. $\lim_{r\to\infty}\limsup_{n\to\infty}\frac{1}{n}\log P((\int_0^1\|u_t^n\|^2\,dt)^{\frac{1}{2}}>r)=-\infty$.

Proof. As

$$
P\left(\left(\int_0^1\left\|u_t^n\right\|^2dt\right)^{\frac{1}{2}}>r\right)=P\left(\left(\int_0^1\left\|nu_t^n\right\|^2dt\right)^{\frac{1}{2}}>nr\right)=P\left(\left(\int_0^1\left\|X_t^n\right\|^2dt\right)^{\frac{1}{2}}>nr\right),
$$

and

$$\left(\int_0^1 \|X_t^n\|^2\,dt\right)^{\frac{1}{2}} = \left(\int_0^1 \left(1+\lambda\left|X_t^n\right|^2\right)^{-\frac{1}{2}}\cdot\|X_t^n\|^2\cdot\left(1+\lambda\left|X_t^n\right|^2\right)^{\frac{1}{2}}\,dt\right)^{\frac{1}{2}}$$
$$\leqslant \left(\sup_{0\leqslant t\leqslant 1}\left(1+\lambda\left|X_t^n\right|^2\right)^{\frac{1}{2}}\right)^{\frac{1}{2}}\cdot\left(\int_0^1 \left(1+\lambda\left|X_t^n\right|^2\right)^{-\frac{1}{2}}\cdot\|X_t^n\|^2\,dt\right)^{\frac{1}{2}}$$
$$\leqslant \frac{1}{2}\sup_{0\leqslant t\leqslant 1}\left(1+\lambda\left|X_t^n\right|^2\right)^{\frac{1}{2}} + \frac{1}{2}\int_0^1 \left(1+\lambda\left|X_t^n\right|^2\right)^{-\frac{1}{2}}\cdot\|X_t^n\|^2\,dt,$$

it is sufficient to show

$$\lim_{r\to\infty}\limsup_{n\to\infty}\frac{1}{n}\log P\left(\sup_{0\leqslant t\leqslant 1}\left(1+\lambda\left|X_t^n\right|^2\right)^{\frac{1}{2}} > nr\right) = -\infty, \tag{3.8}$$

and

$$\lim_{r\to\infty}\limsup_{n\to\infty}\frac{1}{n}\log P\left(\int_0^1 \left(1+\lambda\left|X_t^n\right|^2\right)^{-\frac{1}{2}}\cdot\|X_t^n\|^2\,dt > nr\right) = -\infty. \tag{3.9}$$

(3.8) follows from the previous lemma. We prove (3.9).

Set $Y_1^n := \int_0^1 (1+\lambda|X_t^n|^2)^{-\frac{1}{2}}\cdot\|X_t^n\|^2\,dt$. Then

$$\lambda\nu\left(1+\lambda\left|X_t^n\right|^2\right)^{-\frac{1}{2}}\cdot\|X_t^n\|^2 = \left\langle AX_t^n + \frac{1}{n}B\left(X_t^n\right), g'\left(X_t^n\right)\right\rangle,$$

and

$$\int_0^1 h\left(X_s^n\right)ds + \lambda\nu Y_1^n \leqslant n|\mathrm{b}|\lambda^{\frac{1}{2}} + n\lambda\,Tr\,Q + nM_\lambda.$$

Therefore,

$$\begin{aligned}P\left(Y_1^n > nr\right) &= P\left(\lambda\nu Y_1^n > \lambda\nu nr\right)\\ &\leqslant P\left(g\left(X_1^n\right) + \lambda\nu Y_1^n > \lambda\nu nr\right)\\ &= P\left(\left(g\left(X_1^n\right) - g(nx) - \int_0^1 h\left(X_s^n\right)ds + g(nx) + \int_0^1 h\left(X_s^n\right)ds\right) + \lambda\nu Y_1^n > \lambda\nu nr\right)\end{aligned}$$

$$\leqslant P\left(\left(g\left(X_1^n\right) - g(nx) - \int_0^1 h\left(X_s^n\right) ds\right) > \lambda \nu n r - \left(1 + \lambda n^2 |x|^2\right)^{\frac{1}{2}} \right.$$
$$\left. - n\left(|\mathrm{b}|\lambda^{\frac{1}{2}} + \lambda \, Tr\, Q + M_\lambda\right)\right)$$
$$\leqslant \exp\left(-\lambda \nu n r + \left(1 + \lambda n^2 |x|^2\right)^{\frac{1}{2}} + n\left(|\mathrm{b}|\lambda^{\frac{1}{2}} + \lambda \, Tr\, Q + M_\lambda\right)\right), \tag{3.10}$$

which yields

$$\lim_{r\to\infty} \limsup_{n\to\infty} \frac{1}{n} \log P\left(Y_1^n > nr\right) = -\infty,$$

completing the proof. □

Define the projection operator P_m by

$$P_m x := \sum_{i=1}^m (x, e_i) e_i, \quad x \in H.$$

Let $Z_t^{n,m}$, Z_t^n be the solutions of the following linear equations respectively,

$$Z_t^{n,m} = -\int_0^t A Z_s^{n,m} \, ds + \frac{1}{n} \int_0^t \int_X P_m f(x) \tilde{N}_n(ds, dx), \tag{3.11}$$

and

$$Z_t^n = -\int_0^t A Z_s^n \, ds + \frac{1}{n} \int_0^t \int_X f(x) \tilde{N}_n(ds, dx). \tag{3.12}$$

Put $\tilde{Z}_t^{n,m} := n(Z_t^{n,m} - Z_t^n)$, then $\tilde{Z}_t^{n,m}$ is the solution of the equation

$$\tilde{Z}_t^{n,m} = -\int_0^t A \tilde{Z}_s^{n,m} \, ds + \int_0^t \int_X \left(P_m f(x) - f(x)\right) \tilde{N}_n(ds, dx). \tag{3.13}$$

Similar to the proof of Lemma 3.1, one has

$$\exp\left(g\left(\tilde{Z}_t^{n,m}\right) - g(0) - \int_0^t \tilde{h}\left(\tilde{Z}_s^{n,m}\right) ds\right) \quad \text{is a } \mathcal{F}_t\text{-local martingale}, \tag{3.14}$$

where

$$\tilde{h}(y) = -\langle Ay, g'(y)\rangle + n\int_X \big\{\exp\big[g\big(y + P_m f(x) - f(x)\big) - g(y)\big] - 1$$
$$- \big(g'(y), P_m f(x) - f(x)\big)\big\}\nu(dx).$$

Lemma 3.4. *For any $\delta > 0$,*

$$\lim_{m\to\infty}\limsup_{n\to\infty}\frac{1}{n}\log P\left(\int_0^1 \big\|Z_s^{n,m} - Z_s^n\big\|^2\,ds > \delta\right) = -\infty.$$

Proof. By Lemma 5.6 in [16], we know that, for any $\delta > 0$,

$$\lim_{m\to\infty}\limsup_{n\to\infty}\frac{1}{n}\log P\Big(\sup_{0\leqslant t\leqslant 1}\big|Z_t^{n,m} - Z_t^n\big| > \delta\Big) = -\infty. \tag{3.15}$$

Define a stopping time by

$$\tau_1^{n,m} := \inf\big\{t \geqslant 0,\ \big|Z_t^{n,m} - Z_t^n\big| > 1\big\},$$

then

$$P\left(\int_0^1 \big\|Z_s^{n,m} - Z_s^n\big\|^2\,ds > \delta,\ \sup_{0\leqslant t\leqslant 1}\big|Z_t^{n,m} - Z_t^n\big| \leqslant 1\right)$$
$$\leqslant P\left(\int_0^1 \big\|Z_s^{n,m} - Z_s^n\big\|^2\,ds > \delta,\ 1 \leqslant \tau_1^{n,m}\right)$$
$$\leqslant P\left(\int_0^{1\wedge\tau_1^{n,m}} \big\|Z_s^{n,m} - Z_s^n\big\|^2\,ds > \delta\right)$$
$$\leqslant P\Bigg(\sup_{0\leqslant t\leqslant 1\wedge\tau_1^{n,m}}\big(g\big(nZ_{t-}^{n,m} - nZ_{t-}^n\big)\big)$$
$$\times \int_0^{1\wedge\tau_1^{n,m}} g^{-1}\big(nZ_s^{n,m} - nZ_s^n\big)\cdot\big\|nZ_s^{n,m} - nZ_s^n\big\|^2\,ds > n^2\delta\Bigg),$$

where in the last step, we have used the fact that $\{s\colon nZ_{s-}^{n,m} - nZ_{s-}^n \neq nZ_s^{n,m} - nZ_s^n\}$ is countable. By the definition of g and the stopping time $\tau_1^{n,m}$, we get

$$P\Bigg(\sup_{0\leqslant t\leqslant 1\wedge\tau_1^{n,m}}\big(g\big(nZ_{t-}^{n,m}-nZ_{t-}^n\big)\big)\cdot\int_0^{1\wedge\tau_1^{n,m}} g^{-1}\big(nZ_s^{n,m}-nZ_s^n\big)\cdot\big\|nZ_s^{n,m}-nZ_s^n\big\|^2\,ds>n^2\delta\Bigg)$$

$$\leqslant P\Bigg(\int_0^{1\wedge\tau_1^{n,m}} g^{-1}\big(nZ_s^{n,m}-nZ_s^n\big)\cdot\big\|nZ_s^{n,m}-nZ_s^n\big\|^2\,ds>\frac{n^2\delta}{(1+\lambda n^2)^{\frac{1}{2}}}\Bigg).$$

Set $\hat{Z}_t^{n,m}:=\int_0^t g^{-1}(\tilde{Z}_s^{n,m})\cdot\|\tilde{Z}_s^{n,m}\|^2\,ds$, then

$$\int_0^{1\wedge\tau_1^{n,m}}\tilde{h}\big(\tilde{Z}_t^{n,m}\big)+\lambda\nu\hat{Z}_{1\wedge\tau_1^{n,m}}^{n,m}\leqslant nM_{\lambda,m},\tag{3.16}$$

where

$$M_{\lambda,m}=\lambda\int_X\exp\Big(\lambda^{\frac{1}{2}}\big|P_mf(x)-f(x)\big|\Big)\cdot\big(\big|P_mf(x)-f(x)\big|^2\big)\nu(dx).$$

As in the proof of (3.10), we have

$$\begin{aligned}
&P\bigg(\hat{Z}_{1\wedge\tau_1^{n,m}}^{n,m}>\frac{n^2\delta}{(1+\lambda n^2)^{\frac{1}{2}}}\bigg)\\
&\quad=P\bigg(\lambda\nu\hat{Z}_{1\wedge\tau_1^{n,m}}^{n,m}>\frac{\lambda\nu n^2\delta}{(1+\lambda n^2)^{\frac{1}{2}}}\bigg)\\
&\quad\leqslant P\bigg(g\big(\tilde{Z}_{1\wedge\tau_1^{n,m}}^{n,m}\big)+\lambda\nu\hat{Z}_{1\wedge\tau_1^{n,m}}^{n,m}>\frac{\lambda\nu n^2\delta}{(1+\lambda n^2)^{\frac{1}{2}}}\bigg)\\
&\quad\leqslant P\Bigg(\Bigg[g\big(\tilde{Z}_{1\wedge\tau_1^{n,m}}^{n,m}\big)-g(0)-\int_0^{1\wedge\tau_1^{n,m}}\tilde{h}\big(\tilde{Z}_s^{n,m}\big)\,ds+g(0)+\int_0^{1\wedge\tau_1^{n,m}}\tilde{h}\big(\tilde{Z}_s^{n,m}\big)\,ds\Bigg]+\lambda\nu\hat{Z}_{1\wedge\tau_1^{n,m}}^{n,m}\\
&\qquad>\frac{\lambda\nu n^2\delta}{(1+\lambda n^2)^{\frac{1}{2}}}\Bigg)\\
&\quad\leqslant P\Bigg(g\big(\tilde{Z}_{1\wedge\tau_1^{n,m}}^{n,m}\big)-g(0)-\int_0^{1\wedge\tau_1^{n,m}}\tilde{h}\big(\tilde{Z}_s^{n,m}\big)\,ds\Bigg)>\frac{\lambda\nu n^2\delta}{(1+\lambda n^2)^{\frac{1}{2}}}-g(0)-nM_{\lambda,m})\\
&\quad\leqslant\exp\bigg(-\frac{\lambda\nu n^2\delta}{(1+\lambda n^2)^{\frac{1}{2}}}+1+nM_{\lambda,m}\bigg),
\end{aligned}$$

where we used the fact in (3.14). Therefore,

$$\limsup_{n\to\infty}\frac{1}{n}\log P\left(\int_0^1 \left\|Z_s^{n,m}-Z_s^n\right\|^2 ds>\delta,\ \sup_{0\leqslant t\leqslant 1}\left|Z_t^{n,m}-Z_t^n\right|\leqslant 1\right)\leqslant -\nu\delta\lambda^{\frac{1}{2}}+M_{\lambda,m}. \quad (3.17)$$

Since $\lim_{m\to\infty} M_{\lambda,m}=0$, let $m\to\infty$ in (3.17) to obtain

$$\lim_{m\to\infty}\limsup_{n\to\infty}\frac{1}{n}\log P\left(\int_0^1 \left\|Z_s^{n,m}-Z_s^n\right\|^2 ds>\delta,\ \sup_{0\leqslant t\leqslant 1}\left|Z_t^{n,m}-Z_t^n\right|\leqslant 1\right)\leqslant -\nu\delta\lambda^{\frac{1}{2}}.$$

Since λ is arbitrary, taking $\lambda\to\infty$ in the above inequality, we obtain

$$\lim_{m\to\infty}\limsup_{n\to\infty}\frac{1}{n}\log P\left(\int_0^1 \left\|Z_s^{n,m}-Z_s^n\right\|^2 ds>\delta,\ \sup_{0\leqslant t\leqslant 1}\left|Z_t^{n,m}-Z_t^n\right|\leqslant 1\right)=-\infty.$$

Combining with (3.15) proves the lemma. □

4. Large deviation principle

First, we state the main result of this paper. For $l\in H$, define

$$F(l):=\int_X\left[\exp\bigl(f(x),l\bigr)-1-\bigl(f(x),l\bigr)\right]\nu(dx)+(Ql,l)+(\mathrm{b},l).$$

Set, for $z\in H$,

$$F^*(z)=\sup_{l\in H}\bigl[(z,l)-F(l)\bigr].$$

Let

$$L_t^n:=\mathrm{b}t+\frac{1}{\sqrt{n}}W_t+\frac{1}{n}\int_0^t\int_X f(x)\tilde{N}_n(ds,dx),$$

then by [7], we know that the laws of $\{L_\cdot^n, n\geqslant 1\}$ satisfy a large deviation principle on $D([0,1];H)$ with the rate function I_0, which is defined by

$$I_0(g):=\begin{cases}\int_0^1 F^*(g'(s))\,ds, & \text{if } g\in D([0,1];H),\ g'\in L^1([0,1];H),\\ \infty, & \text{otherwise.}\end{cases}$$

For $g\in D([0,1];H)$ with $g'\in L^1([0,1];H)$, define $\phi(g)\in D([0,1];H)\cap L^2([0,1];V)$ to be the solution of the following equation

$$\phi_t(g)=x-\int_0^t A\phi_s(g)\,ds-\int_0^t B\bigl(\phi_s(g)\bigr)\,ds+g(t). \quad (4.1)$$

Set, for $h \in D([0,1];H)$,

$$I(h) := \inf\bigl\{I_0(g):\ h = \phi(g),\ g \in D\bigl([0,1];H\bigr)\bigr\}$$

with the convention $\inf\{\emptyset\} = \infty$.

Theorem 4.1. *Let μ_n be the law of the solution u^n of Eq.* (1.3)*, then $\{\mu_n, n \geqslant 1\}$ satisfies a large deviation principle on $D([0,1];H)$ endowed with the uniform topology with the rate function $I(\cdot)$, i.e.,*

(i) *For any closed subset $F \subset D([0,1];H)$,*

$$\limsup_{n\to\infty} \frac{1}{n} \log \mu_n(F) \leqslant -\inf_{h\in F} I(h).$$

(ii) *For any open set $G \subset D([0,1];H)$,*

$$\liminf_{n\to\infty} \frac{1}{n} \log \mu_n(G) \geqslant -\inf_{h\in G} I(h).$$

According to the generalized contraction principle in the theory of large deviations (see Theorem 4.1 in [9]), Theorem 4.1 will follow from the following Lemmas 4.1–4.3.

Lemma 4.1. *The mapping ϕ defined in* (4.1) *is continuous from $D([0,1];V)$ into $D([0,1];H) \cap L^2([0,1];V)$ in the topology of uniform convergence.*

Proof. Let $v_t(g) = \phi_t(g) - g(t)$, then $v_t(g)$ satisfies the following equation

$$v_t(g) = x - \int_0^t A v_s(g)\,ds - \int_0^t Ag(s)\,ds - \int_0^t B\bigl(v_s(g) + g(s)\bigr)\,ds.$$

It is sufficient to show that

$$v(\cdot) : D\bigl([0,1];V\bigr) \to D\bigl([0,1];H\bigr) \cap L^2\bigl([0,1];V\bigr) \quad \text{is continuous,}$$

that is, take $\{g_n\}_{n=1}^{\infty},\ g \in D([0,1];V)$ such that $\lim_{n\to\infty} \sup_{0\leqslant t\leqslant 1} \|g_n(t) - g(t)\| = 0$, then

$$\lim_{n\to\infty} \left(\sup_{0\leqslant t\leqslant 1} \bigl|v_t(g) - v_t(g_n)\bigr|^2 + \frac{\nu}{2} \int_0^1 \bigl\|v_s(g) - v_s(g_n)\bigr\|^2\,ds \right) = 0.$$

To this end, we need some energy estimates for $v_t(g)$. In view of (2.3), (2.4), (2.5), and (2.9), we have

$$
\begin{aligned}
&\left|v_t(g)\right|^2+2\nu\int_0^t\left\|v_s(g)\right\|^2ds\\
&=|x|^2-2\int_0^t\left\langle Ag(s),v_s(g)\right\rangle ds-2\int_0^t\left\langle B\left(v_s(g)+g(s)\right),v_s(g)\right\rangle ds\\
&\leqslant|x|^2+\nu\int_0^t\left\|v_s(g)\right\|^2ds+\frac{1}{\nu}\int_0^t\left\|g(s)\right\|^2ds+2\int_0^t\left|\left\langle B\left(v_s(g)+g(s)\right),v_s(g)\right\rangle\right|ds\\
&\leqslant|x|^2+\nu\int_0^t\left\|v_s(g)\right\|^2ds+\frac{1}{\nu}\int_0^t\left\|g(s)\right\|^2ds+2\int_0^t\left|b\left(v_s(g),g(s),v_s(g)\right)\right|ds\\
&\quad+2\int_0^t\left|b\left(g(s),g(s),v_s(g)\right)\right|ds\\
&\leqslant|x|^2+\nu\int_0^t\left\|v_s(g)\right\|^2ds+\frac{1}{\nu}\int_0^t\left\|g(s)\right\|^2ds+4\int_0^t\left|v_s(g)\right|\cdot\left\|g(s)\right\|\cdot\left\|v_s(g)\right\|ds\\
&\quad+4c\int_0^t\left\|g(s)\right\|^2\left\|v_s(g)\right\|ds\\
&\leqslant|x|^2+\frac{3\nu}{2}\int_0^t\left\|v_s(g)\right\|^2ds+\frac{1}{\nu}\int_0^t\left\|g(s)\right\|^2ds+\frac{16}{\nu}\int_0^t\left|v_s(g)\right|^2\left\|g(s)\right\|^2ds\\
&\quad+\frac{16c^2}{\nu}\int_0^t\left\|g(s)\right\|^4ds. \qquad (4.2)
\end{aligned}
$$

Applying Gronwall's inequality, we have

$$
\begin{aligned}
\sup_{0\leqslant s\leqslant t}\left|v_s(g)\right|^2&\leqslant\left(|x|^2+\frac{1}{\nu}\int_0^t\left\|g(s)\right\|^2ds+\frac{16c^2}{\nu}\int_0^t\left\|g(s)\right\|^4ds\right)\exp\left(\frac{16}{\nu}\int_0^t\left\|g(s)\right\|^2ds\right)\\
&\leqslant\left[|x|^2+t\left(\frac{1}{\nu}\sup_{0\leqslant s\leqslant t}\left\|g(s)\right\|^2+\frac{16c^2}{\nu}\sup_{0\leqslant s\leqslant t}\left\|g(s)\right\|^4\right)\right]\exp\left(\frac{16}{\nu}t\sup_{0\leqslant s\leqslant t}\left\|g(s)\right\|^2\right).
\end{aligned}
\qquad (4.3)
$$

Furthermore,

$$\frac{\nu}{2}\int_0^t \|v_s(g)\|^2\,ds$$
$$\leqslant |x|^2 + \frac{1}{\nu}t\Big(\sup_{0\leqslant s\leqslant t}\|g(s)\|^2 + 16\sup_{0\leqslant s\leqslant t}\|g(s)\|^2 M_t(g) + 16c^2\sup_{0\leqslant s\leqslant t}\|g(s)\|^4\Big), \tag{4.4}$$

where

$$M_g(t) = \left[|x|^2 + t\left(\frac{1}{\nu}\sup_{0\leqslant s\leqslant t}\|g(s)\|^2 + \frac{16c^2}{\nu}\sup_{0\leqslant s\leqslant t}\|g(s)\|^4\right)\right]\exp\left(\frac{16}{\nu}t\sup_{0\leqslant s\leqslant t}\|g(s)\|^2\right).$$

It is easy to see that, if g is replaced by g_n, (4.3), (4.4) still hold. Since

$$\lim_{n\to\infty}\sup_{0\leqslant t\leqslant 1}\|g_n(t)-g(t)\|^2 = 0,$$

there exists a constant $C_g(x)$ depending on $\sup_{0\leqslant t\leqslant 1}\|g(t)\|$ and $|x|^2$ (which may change line to line) such that

$$\int_0^1 \|v_s(g_n)\|^2\,ds \leqslant C_g(x) \quad (n\geqslant 1). \tag{4.5}$$

Now by the chain rule, one has

$$\begin{aligned}
&\big|v_t(g_n)-v_t(g)\big|^2 + 2\nu\int_0^t \|v_s(g_n)-v_s(g)\|^2\,ds \\
&\quad= -2\int_0^t \big\langle Ag_n(s)-Ag(s), v_s(g_n)-v_s(g)\big\rangle\,ds \\
&\qquad - 2\int_0^t \big\langle B\big(v_s(g_n)+g_n(s)\big) - B\big(v_s(g)+g(s)\big), v_s(g_n)-v_s(g)\big\rangle\,ds \\
&\quad\leqslant \nu\int_0^t \|v_s(g_n)-v_s(g)\|^2\,ds + \frac{1}{\nu}\int_0^t \|g_n(s)-g(s)\|^2\,ds \\
&\qquad + 2\int_0^t \big|\big\langle B\big(v_s(g_n)+g_n(s)\big) - B\big(v_s(g)+g(s)\big), v_s(g_n)-v_s(g)\big\rangle\big|\,ds.
\end{aligned} \tag{4.6}$$

Now, due to (2.3), (2.4), (2.5) and (2.9),

$$\begin{aligned}
&\big|\big\langle B\big(v_s(g_n)+g_n(s)\big)-B\big(v_s(g)+g(s)\big), v_s(g_n)-v_s(g)\big\rangle\big| \\
&\quad\leqslant \big|b\big(v_s(g_n)+g_n(s), v_s(g_n)+g_n(s), v_s(g_n)-v_s(g)\big) \\
&\qquad - b\big(v_s(g)+g(s), v_s(g_n)+g_n(s), v_s(g_n)-v_s(g)\big)\big| \\
&\qquad + \big|b\big(v_s(g)+g(s), v_s(g)+g(s), v_s(g_n)-v_s(g)\big) \\
&\qquad - b\big(v_s(g)+g(s), v_s(g_n)+g_n(s), v_s(g_n)-v_s(g)\big)\big| \\
&\quad\leqslant \big|b\big(v_s(g_n)-v_s(g), v_s(g_n)+g_n(s), v_s(g_n)-v_s(g)\big)\big| \\
&\qquad + \big|b\big(g_n(s)-g(s), v_s(g_n)+g_n(s), v_s(g_n)-v_s(g)\big)\big| \\
&\qquad + \big|b\big(v_s(g)+g(s), g_n(s)-g(s), v_s(g_n)-v_s(g)\big)\big| \\
&\quad\leqslant 2\big|v_s(g_n)-v_s(g)\big|\cdot\big\|v_s(g_n)-v_s(g)\big\|\cdot\big\|v_s(g_n)+g_n(s)\big\| \\
&\qquad + 2c\big\|g_n(s)-g(s)\big\|\cdot\big\|v_s(g_n)+g_n(s)\big\|\cdot\big\|v_s(g_n)-v_s(g)\big\| \\
&\qquad + 2c\big\|v_s(g)+g(s)\big)|\cdot\big\|g_n(s)-g(s)\big\|\cdot\big\|v_s(g_n)-v_s(g)\big\| \\
&\quad\leqslant \frac{\nu}{12}\big\|v_s(g_n)-v_s(g)\big\|^2+\frac{12}{\nu}\big(\big\|v_s(g)\big\|+\big\|g_n(s)\big\|\big)^2\cdot\big|v_s(g_n)-v_s(g)\big|^2 \\
&\qquad + \frac{\nu}{12}\big\|v_s(g_n)-v_s(g)\big\|^2+\frac{12c^2}{\nu}\big(\big\|v_s(g_n)\big\|+\big\|g_n(s)\big\|\big)^2\cdot\big\|g_n(s)-g(s)\big\|^2 \\
&\qquad + \frac{\nu}{12}\big\|v_s(g_n)-v_s(g)\big\|^2+\frac{12c^2}{\nu}\big(\big\|v_s(g)\big\|+\big\|g(s)\big\|\big)^2\cdot\big\|g_n(s)-g(s)\big\|^2.
\end{aligned} \tag{4.7}$$

Combining (4.6) and (4.7), one obtains

$$\begin{aligned}
&\big|v_t(g_n)-v_t(g)\big|^2+\frac{\nu}{2}\int_0^t\big\|v_s(g_n)-v_s(g)\big\|^2\,ds \\
&\quad\leqslant \frac{1}{\nu}\int_0^t\big\|g_n(s)-g(s)\big\|^2\,ds+\frac{24}{\nu}\int_0^t\big(\big\|v_s(g)\big\|+\big\|g_n(s)\big\|\big)^2\cdot\big|v_s(g_n)-v_s(g)\big|^2\,ds \\
&\qquad + \frac{24c^2}{\nu}\int_0^t\big(\big\|v_s(g_n)\big\|+\big\|g_n(s)\big\|\big)^2\cdot\big\|g_n(s)-g(s)\big\|^2\,ds \\
&\qquad + \frac{24c^2}{\nu}\int_0^t\big(\big\|v_s(g)\big\|+\big\|g(s)\big\|\big)^2\cdot\big\|g_n(s)-g(s)\big\|^2\,ds.
\end{aligned}$$

Applying Gronwall's inequality and (4.5), we arrive at

$$\sup_{0\leqslant t\leqslant 1}\left|v_t(g_n)-v_t(g)\right|^2+\frac{\nu}{2}\int_0^1\left\|v_s(g_n)-v_s(g)\right\|^2 ds$$

$$\leqslant\left(\frac{48c^2}{\nu}\int_0^1\left[\left(\left\|v_s(g_n)\right\|+\left\|g_n(s)\right\|\right)^2+\left(\left\|v_s(g)\right\|+\left\|g(s)\right\|\right)^2\right]\cdot\left\|g_n(s)-g(s)\right\|^2 ds\right.$$

$$\left.+\frac{2}{\nu}\int_0^1\left\|g_n(s)-g(s)\right\|^2 ds\right)\times\exp\left(\frac{48}{\nu}\int_0^1\left(\left\|v_s(g_n)\right\|+\left\|g_n(s)\right\|\right)^2 ds\right)$$

$$\leqslant\left(\frac{2}{\nu}+\frac{48c^2}{\nu}\right)C_g(x)\sup_{0\leqslant t\leqslant 1}\left\|g_n(t)-g(t)\right\|^2.$$

Let $n\to\infty$ to prove the lemma. □

Now, let $u_\cdot^{n,m}$ be the solution of the equation

$$u_t^{n,m}=x-\int_0^t Au_s^{n,m}\,ds-\int_0^t B\left(u_s^{n,m}\right)ds+\mathrm{b}^m t+\frac{1}{\sqrt{n}}W_t^m+\frac{1}{n}\int_0^t\int_X f^m(x)\tilde{N}_n(ds,dx), \quad (4.8)$$

where $\mathrm{b}^m=P_m\mathrm{b}$, $W_t^m=P_mW_t$, $f^m(x)=P_mf(x)$. Recall that $Z^{n,m}$, Z^n are defined as in (3.11) and (3.12). Set $\bar{u}_t^{n,m}:=u_t^{n,m}-Z_t^{n,m}$, $\bar{u}_t^n:=u_t^n-Z_t^n$. Then $\bar{u}_t^{n,m}$ and $\bar{u}_t^n$ satisfy

$$\bar{u}_t^{n,m}=x-\int_0^t A\bar{u}_s^{n,m}\,ds-\int_0^t B\left(\bar{u}_s^{n,m}+Z_s^{n,m}\right)ds+\mathrm{b}^m t+\frac{1}{\sqrt{n}}W_t^m,$$

and

$$\bar{u}_t^n=x-\int_0^t A\bar{u}_s^n\,ds-\int_0^t B\left(\bar{u}_s^n+Z_s^n\right)ds+\mathrm{b}t+\frac{1}{\sqrt{n}}W_t.$$

Lemma 4.2. *For any $\delta>0$,*

$$\lim_{m\to\infty}\limsup_{n\to\infty}\frac{1}{n}\log P\left(\sup_{0\leqslant t\leqslant 1}\left|u_t^{n,m}-u_t^n\right|>\delta\right)=-\infty.$$

Proof. Note that

$$P\left(\sup_{0\leqslant t\leqslant 1}\left|u_t^{n,m}-u_t^n\right|>\delta\right)$$

$$\leqslant P\left(\sup_{0\leqslant t\leqslant 1}\left|\bar{u}_t^{n,m}-\bar{u}_t^n\right|>\frac{\delta}{2}\right)+P\left(\sup_{0\leqslant t\leqslant 1}\left|Z_t^{n,m}-Z_t^n\right|>\frac{\delta}{2}\right). \quad (4.9)$$

By Lemma 5.6 in [16], we know that

$$\lim_{m\to\infty}\limsup_{n\to\infty}\frac{1}{n}\log P\left(\sup_{0\leqslant t\leqslant 1}\left|Z_t^{n,m}-Z_t^n\right|>\frac{\delta}{2}\right)=-\infty. \tag{4.10}$$

It suffices to prove, for any $\delta>0$,

$$\lim_{m\to\infty}\limsup_{n\to\infty}\frac{1}{n}\log P\left(\sup_{0\leqslant t\leqslant 1}\left|\bar{u}_t^{n,m}-\bar{u}_t^n\right|>\frac{\delta}{2}\right)=-\infty. \tag{4.11}$$

For $\delta_0>0$, define a stopping time by

$$\tau_{\delta_0}^{n,m}:=\inf\left\{t\geqslant 0,\ \left|Z_t^{n,m}-Z_t^n\right|>\delta_0 \text{ or } \int_0^t \left\|Z_s^{n,m}-Z_s^n\right\|^2 ds>\delta_0\right\}.$$

Note that Lemma 3.2 and Lemma 3.3 still hold for $u_\cdot^{n,m}$, $Z_\cdot^{n,m}$ and $Z_\cdot^n$, when m is fixed. Define stopping times $\tau_{u,1,M}^n$, $\tau_{u,2,M}^n$ by

$$\tau_{u,1,M}^n:=\inf\left\{t\geqslant 0,\ \left|u^n(t)\right|>M\right\},$$

and

$$\tau_{u,2,M}^n:=\inf\left\{t\geqslant 0,\ \int_0^t \left\|u_s^n\right\|^2 ds>M\right\}.$$

We can also define similar stopping times for $u_\cdot^{n,m}$, $Z_\cdot^{n,m}$ and $Z_\cdot^n$. We denote these stopping times by $\tau_{u,1,M}^{n,m}$, $\tau_{u,2,M}^{n,m}$, $\tau_{Z,1,M}^n$, $\tau_{Z,2,M}^n$, $\tau_{Z,1,M}^{n,m}$ and $\tau_{Z,2,M}^{n,m}$, respectively. Let

$$\tau_M^{n,m}:=\tau_{u,1,M}^n\wedge\tau_{u,2,M}^n\wedge\tau_{u,1,M}^{n,m}\wedge\tau_{u,2,M}^{n,m}\wedge\tau_{Z,1,M}^n\wedge\tau_{Z,2,M}^n\wedge\tau_{Z,1,M}^{n,m}\wedge\tau_{Z,2,M}^{n,m},$$

and set

$$\begin{aligned}A^{n,m}(\omega):={}&\left\{\sup_{0\leqslant t\leqslant 1}\left|u_t^{n,m}\right|\leqslant M\right\}\cap\left\{\sup_{0\leqslant t\leqslant 1}\left|u_t^n\right|\leqslant M\right\}\cap\left\{\sup_{0\leqslant t\leqslant 1}\left|Z_t^{n,m}\right|\leqslant M\right\}\\&\cap\left\{\sup_{0\leqslant t\leqslant 1}\left|Z_t^n\right|\leqslant M\right\},\\B^{n,m}(\omega):={}&\left\{\int_0^1\left\|u_s^{n,m}\right\|^2 ds\leqslant M\right\}\cap\left\{\int_0^1\left\|u_s^n\right\|^2 ds\leqslant M\right\}\cap\left\{\int_0^1\left\|Z_s^{n,m}\right\|^2 ds\leqslant M\right\}\\&\cap\left\{\int_0^1\left\|Z_s^n\right\|^2 ds\leqslant M\right\},\end{aligned}$$

$$C^{n,m}(\omega) := \left\{ \sup_{0\leqslant t\leqslant 1} \left|Z_t^{n,m} - Z_t^n\right| \leqslant \delta_0,\ \int_0^1 \left\|Z_t^{n,m} - Z_t^n\right\|^2 dt \leqslant \delta_0 \right\}.$$

Then,

$$\begin{aligned} &P\Big(\Big\{\sup_{0\leqslant t\leqslant 1} \left|\bar u_t^{n,m} - \bar u_t^n\right| > \delta\Big\} \cap A^{n,m} \cap B^{n,m} \cap C^{n,m}\Big) \\ &\quad \leqslant P\Big(\sup_{0\leqslant t\leqslant 1} \left|\bar u_t^{n,m} - \bar u_t^n\right| > \delta, 1 \leqslant \tau_M^{n,m} \wedge \tau_{\delta_0}^{n,m}\Big) \\ &\quad \leqslant P\Big(\sup_{0\leqslant t\leqslant 1\wedge\tau_M^{n,m}\wedge\tau_{\delta_0}^{n,m}} \left|\bar u_t^{n,m} - \bar u_t^n\right| > \delta\Big) \\ &\quad = P\Big(\sup_{0\leqslant t\leqslant 1\wedge\tau_M^{n,m}\wedge\tau_{\delta_0}^{n,m}} \left|\bar u_t^{n,m} - \bar u_t^n\right|^2 > \delta^2\Big). \end{aligned} \tag{4.12}$$

Applying Itô's formula to $|\bar u^{n,m}_{t\wedge\tau_M^{n,m}\wedge\tau_{\delta_0}^{n,m}} - \bar u^n_{t\wedge\tau_M^{n,m}\wedge\tau_{\delta_0}^{n,m}}|^2$, we have

$$\begin{aligned} &\left|\bar u^{n,m}_{t\wedge\tau_M^{n,m}\wedge\tau_{\delta_0}^{n,m}} - \bar u^n_{t\wedge\tau_M^{n,m}\wedge\tau_{\delta_0}^{n,m}}\right|^2 + 2\nu \int_0^{t\wedge\tau_M^{n,m}\wedge\tau_{\delta_0}^{n,m}} \left\|\bar u_s^{n,m} - \bar u_s^n\right\|^2 ds \\ &= -2 \int_0^{t\wedge\tau_M^{n,m}\wedge\tau_{\delta_0}^{n,m}} \left\langle B\left(\bar u_s^{n,m} + Z_s^{n,m}\right) - B\left(\bar u_s^n + Z_s^n\right), \bar u_s^{n,m} - \bar u_s^n\right\rangle ds \\ &\quad + 2 \int_0^{t\wedge\tau_M^{n,m}\wedge\tau_{\delta_0}^{n,m}} \left(\bar u_s^{n,m} - \bar u_s^n, \mathrm{b}^m - \mathrm{b}\right) ds + \frac{1}{n} \int_0^{t\wedge\tau_M^{n,m}\wedge\tau_{\delta_0}^{n,m}} \sum_{i=m+1}^{\infty} \lambda_i\, ds \\ &\quad + \frac{2}{\sqrt{n}} \int_0^{t\wedge\tau_M^{n,m}\wedge\tau_{\delta_0}^{n,m}} \left(\bar u_s^{n,m} - \bar u_s^n, dW_s^m - dW_s\right), \end{aligned} \tag{4.13}$$

thus,

$$\begin{aligned} &\sup_{0\leqslant s\leqslant t\wedge\tau_M^{n,m}\wedge\tau_{\delta_0}^{n,m}} \left|\bar u_s^{n,m} - \bar u_s^n\right|^2 + 2\nu \int_0^{t\wedge\tau_m^{n,m}\wedge\tau_{\delta_0}^{n,m}} \left\|\bar u_s^{n,m} - \bar u_s^n\right\|^2 ds \\ &\quad \leqslant 4 \int_0^{t\wedge\tau_M^{n,m}\wedge\tau_{\delta_0}^{n,m}} \left|\left\langle B\left(\bar u_s^{n,m} + Z_s^{n,m}\right) - B\left(\bar u_s^n + Z_s^n\right), \bar u_s^{n,m} - \bar u_s^n\right\rangle\right| ds \end{aligned}$$

$$+4\int_0^{t\wedge\tau_M^{n,m}\wedge\tau_{\delta_0}^{n,m}}\left|\left(\bar u_s^{n,m}-\bar u_s^n, b^m-b\right)\right|ds+\frac{2}{n}\int_0^t\sum_{i=m+1}^{\infty}\lambda_i\,ds$$

$$+\frac{4}{\sqrt{n}}\sup_{0\leqslant s\leqslant t\wedge\tau_M^{n,m}\wedge\tau_{\delta_0}^{n,m}}\left|\int_0^s\left(\bar u_r^{n,m}-\bar u_r^n, dW_r^m-dW_r\right)\right|. \tag{4.14}$$

Write,

$$\int_0^t\left\langle B\left(\bar u_s^{n,m}+Z_s^{n,m}\right)-B\left(\bar u_s^n+Z_s^n\right),\bar u_s^{n,m}-\bar u_s^n\right\rangle ds$$

$$=\int_0^t b\left(\bar u_s^{n,m}+Z_s^{n,m},\bar u_s^{n,m}+Z_s^{n,m},\bar u_s^{n,m}-\bar u_s^n\right)-b\left(\bar u_s^n+Z_s^n,\bar u_s^n+Z_s^n,\bar u_s^{n,m}-\bar u_s^n\right)ds$$

$$=\int_0^t b\left(\bar u_s^{n,m},\bar u_s^{n,m},\bar u_s^{n,m}-\bar u_s^n\right)-b\left(\bar u_s^n,\bar u_s^n,\bar u_s^{n,m}-\bar u_s^n\right)ds$$

$$+\int_0^t b\left(\bar u_s^{n,m},Z_s^{n,m},\bar u_s^{n,m}-\bar u_s^n\right)-b\left(\bar u_s^n,Z_s^n,\bar u_s^{n,m}-\bar u_s^n\right)ds$$

$$+\int_0^t b\left(Z_s^{n,m},\bar u_s^{n,m},\bar u_s^{n,m}-\bar u_s^n\right)-b\left(Z_s^n,\bar u_s^n,\bar u_s^{n,m}-\bar u_s^n\right)ds$$

$$+\int_0^t b\left(Z_s^{n,m},Z_s^{n,m},\bar u_s^{n,m}-\bar u_s^n\right)-b\left(Z_s^n,Z_s^n,\bar u_s^{n,m}-\bar u_s^n\right)ds$$

$$=: II_1+II_2+II_3+II_4. \tag{4.15}$$

By virtue of the properties of $b(\cdot\,,\,\cdot\,,\cdot)$, we have,

$$|II_1|\leqslant 2\int_0^t\left|\bar u_s^{n,m}-\bar u_s^n\right|\cdot\left\|\bar u_s^n\right\|\cdot\left\|\bar u_s^{n,m}-\bar u_s^n\right\|ds$$

$$\leqslant\frac{\nu}{24}\int_0^t\left\|\bar u_s^{n,m}-\bar u_s^n\right\|^2ds+\frac{24}{\nu}\int_0^t\left|\bar u_s^{n,m}-\bar u_s^n\right|^2\cdot\left\|\bar u_s^n\right\|^2ds, \tag{4.16}$$

$$|II_2|\leqslant\int_0^t\left|b\left(\bar u_s^{n,m}-\bar u_s^n,Z_s^{n,m},\bar u_s^{n,m}-\bar u_s^n\right)\right|ds+\int_0^t\left|b\left(\bar u_s^n,Z_s^{n,m}-Z_s^n,\bar u_s^{n,m}-\bar u_s^n\right)\right|ds$$

$$\leqslant 2\int_0^t \left|\bar{u}_s^{n,m}-\bar{u}_s^n\right|\cdot\left\|Z_s^{n,m}\right\|\cdot\left\|\bar{u}_s^{n,m}-\bar{u}_s^n\right\|ds$$

$$+2\int_0^t \left|\bar{u}_s^n\right|^{\frac{1}{2}}\cdot\left\|\bar{u}_s^n\right\|^{\frac{1}{2}}\cdot\left|Z_s^{n,m}-Z_s^n\right|^{\frac{1}{2}}\cdot\left\|Z_s^{n,m}-Z_s^n\right\|^{\frac{1}{2}}\cdot\left\|\bar{u}_s^{n,m}-\bar{u}_s^n\right\|ds$$

$$\leqslant \frac{\nu}{24}\int_0^t \left\|\bar{u}_s^{n,m}-\bar{u}_s^n\right\|^2 ds+\frac{24}{\nu}\int_0^t \left|\bar{u}_s^{n,m}-\bar{u}_s^n\right|^2\cdot\left\|Z_s^{n,m}\right\|^2 ds+\frac{\nu}{24}\int_0^t \left\|\bar{u}_s^{n,m}-\bar{u}_s^n\right\|^2 ds$$

$$+\frac{24}{\nu}\sup_{0\leqslant s\leqslant t}\left|Z_{s-}^{n,m}-Z_{s-}^n\right|\cdot\sup_{0\leqslant s\leqslant t}\left|\bar{u}_{s-}^n\right|\cdot\left(\int_0^t \left\|Z_s^{n,m}-Z_s^n\right\|^2 ds\right)^{\frac{1}{2}}\cdot\left(\int_0^t \left\|\bar{u}_s^n\right\|^2 ds\right)^{\frac{1}{2}}, \tag{4.17}$$

$$|II_3|\leqslant 2\int_0^t \left|Z_s^{n,m}-Z_s^n\right|^{\frac{1}{2}}\cdot\left\|Z_s^{n,m}-Z_s^n\right\|^{\frac{1}{2}}\cdot\left|\bar{u}_s^n\right|^{\frac{1}{2}}\cdot\left\|\bar{u}_s^n\right\|^{\frac{1}{2}}\cdot\left\|\bar{u}_s^{n,m}-\bar{u}_s^n\right\|ds$$

$$\leqslant \frac{\nu}{24}\int_0^t \left\|\bar{u}_s^{n,m}-\bar{u}_s^n\right\|^2 ds+\frac{24}{\nu}\sup_{0\leqslant s\leqslant t}\left|Z_{s-}^{n,m}-Z_{s-}^n\right|\cdot\sup_{0\leqslant s\leqslant t}\left|\bar{u}_{s-}^n\right|\left(\int_0^t \left\|Z_s^{n,m}-Z_s^n\right\|^2 ds\right)^{\frac{1}{2}}$$

$$\times\left(\int_0^t \left\|\bar{u}_s^n\right\|^2 ds\right)^{\frac{1}{2}}, \tag{4.18}$$

and similarly

$$|II_4|\leqslant \int_0^t \left|b\left(Z_s^{n,m}-Z_s^n, Z_s^{n,m}, \bar{u}_s^{n,m}-\bar{u}_s^n\right)\right|ds+\int_0^t \left|b\left(Z_s^n, Z_s^{n,m}-Z_s^n, \bar{u}_s^{n,m}-\bar{u}_s^n\right)\right|ds$$

$$\leqslant \frac{\nu}{24}\int_0^t \left\|\bar{u}_s^{n,m}-\bar{u}_s^n\right\|^2 ds+\frac{24}{\nu}\sup_{0\leqslant s\leqslant t}\left|Z_{s-}^{n,m}-Z_{s-}^n\right|\cdot\sup_{0\leqslant s\leqslant t}\left|Z_{s-}^{n,m}\right|\left(\int_0^t \left\|Z_s^{n,m}-Z_s^n\right\|^2 ds\right)^{\frac{1}{2}}$$

$$\times\left(\int_0^t \left\|Z_s^{n,m}\right\|^2 ds\right)^{\frac{1}{2}}$$

$$+\frac{\nu}{24}\int_0^t \left\|\bar{u}_s^{n,m}-\bar{u}_s^n\right\|^2 ds+\frac{24}{\nu}\sup_{0\leqslant s\leqslant t}\left|Z_{s-}^{n,m}-Z_{s-}^n\right|$$

$$\times\sup_{0\leqslant s\leqslant t}\left|Z_{s-}^n\right|\left(\int_0^t \left\|Z_s^{n,m}-Z_s^n\right\|^2 ds\right)^{\frac{1}{2}}\cdot\left(\int_0^t \left\|Z_s^n\right\|^2 ds\right)^{\frac{1}{2}}. \tag{4.19}$$

Note that we have used the fact that $\{s\colon\ Z^n_s \neq Z^n_{s-}\}$ is countable P-a.s. This fact is also true for $Z^{n,m}_\cdot$ and $Z^{n,m}_\cdot - Z^n_\cdot$.

Set $M_t := \frac{4}{\sqrt{n}}\int_0^t (\bar{u}^{n,m}_s - \bar{u}^n_s, dW^m_s - dW_s)$, and putting (4.16)–(4.19) and (4.14) together, one obtains

$$\begin{aligned}
&\frac{1}{2}\sup_{0\leqslant s\leqslant t\wedge\tau^{n,m}_M\wedge\tau^{n,m}_{\delta_0}} \left|\bar{u}^{n,m}_s - \bar{u}^n_s\right|^2 + \nu \int_0^{t\wedge\tau^{n,m}_M\wedge\tau^{n,m}_{\delta_0}} \left\|\bar{u}^{n,m}_s - \bar{u}^n_s\right\|^2 ds \\
&\quad\leqslant \left(\frac{2t}{n}\sum_{i=m+1}^{\infty}\lambda_i + \sup_{0\leqslant s\leqslant t\wedge\tau^{n,m}_M\wedge\tau^{n,m}_{\delta_0}} |M_t| + 8|\mathrm{b}_m - \mathrm{b}|^2 t^2 + \frac{c}{\nu}(\delta_0 M)^{\frac{3}{2}}\right) \\
&\qquad + \frac{c}{\nu}\int_0^{t\wedge\tau^{n,m}_M\wedge\tau^{n,m}_{\delta_0}} \left|\bar{u}^{n,m}_s - \bar{u}^n_s\right|^2 \cdot \left(\left\|\bar{u}^n_s\right\|^2 + \left\|Z^{n,m}_s\right\|^2\right) ds. \qquad (4.20)
\end{aligned}$$

Note that c is a constant independent of n, m. Applying Gronwall's inequality, we get

$$\begin{aligned}
&\sup_{0\leqslant s\leqslant t\wedge\tau^{n,m}_M\wedge\tau^{n,m}_{\delta_0}} \left|\bar{u}^{n,m}_s - \bar{u}^n_s\right|^2 \\
&\quad\leqslant \left(\frac{4t}{n}\sum_{i=m+1}^{\infty}\lambda_i + 2\sup_{0\leqslant s\leqslant t\wedge\tau^{n,m}_M\wedge\tau^{n,m}_{\delta_0}} |M_t| + 16|\mathrm{b}_m - \mathrm{b}|^2 t^2 + \frac{c}{\nu}(\delta_0 M)^{\frac{3}{2}}\right) \\
&\qquad \times \exp^{\frac{c}{\nu}\int_0^{t\wedge\tau^{n,m}_M\wedge\tau^{n,m}_{\delta_0}} (\|\bar{u}^n_s\|^2 + \|Z^{n,m}_s\|^2)\,ds} \\
&\quad\leqslant \left(\frac{4t}{n}\sum_{i=m+1}^{\infty}\lambda_i + 2\sup_{0\leqslant s\leqslant t\wedge\tau^{n,m}_M\wedge\tau^{n,m}_{\delta_0}} |M_t| + 16|\mathrm{b}_m - \mathrm{b}|^2 t^2 + \frac{c}{\nu}(\delta_0 M)^{\frac{3}{2}}\right) \\
&\qquad \times \exp\left(\frac{2Mc}{\nu}\right). \qquad (4.21)
\end{aligned}$$

Set $C_M := \exp(\frac{2Mc}{\nu})$. Applying the martingale inequality in [6] to $M_\cdot$, it follows that

$$\begin{aligned}
&\left[E\Big(\sup_{0\leqslant s\leqslant t\wedge\tau^{n,m}_M\wedge\tau^{n,m}_{\delta_0}} \left|\bar{u}^{n,m}_s - \bar{u}^n_s\right|^{2p}\Big)\right]^{\frac{2}{p}} \\
&\quad\leqslant 2\left(\frac{4t}{n}\sum_{i=m+1}^{\infty}\lambda_i + 16|\mathrm{b}_m - \mathrm{b}|^2 t^2 + \frac{c}{\nu}(\delta_0 M)^{\frac{3}{2}}\right)^2 C_M^2 \\
&\qquad + 8\Big(E\sup_{0\leqslant s\leqslant t\wedge\tau^{n,m}_M\wedge\tau^{n,m}_{\delta_0}} |M_t|^p\Big)^{\frac{2}{p}} C_M^2
\end{aligned}$$

$$\leqslant 2\left(\frac{4t}{n}\sum_{i=m+1}^{\infty}\lambda_i + 16|\mathrm{b}_m - \mathrm{b}|^2 t^2 + \frac{c}{\nu}(\delta_0 M)^{\frac{3}{2}}\right)^2 C_M^2$$

$$+ \frac{cC_M^2}{n}p\left[E\left(\int_0^{t\wedge\tau_M^{n,m}\wedge\tau_{\delta_0}^{n,m}}\sum_{i=m+1}^{\infty}\lambda_i^2\left|\bar{u}_s^{n,m} - \bar{u}_s^n\right|^2 ds\right)^{\frac{p}{2}}\right]^{\frac{2}{p}}$$

$$\leqslant 2\left(\frac{4t}{n}\sum_{i=m+1}^{\infty}\lambda_i + 16|\mathrm{b}_m - \mathrm{b}|^2 t^2 + \frac{c}{\nu}(\delta_0 M)^{\frac{3}{2}}\right)^2 C_M^2 + \frac{cC_M^2}{n}p\left(\sum_{i=m+1}^{\infty}\lambda_i^2\right)^2 t$$

$$+ \frac{cC_M^2}{n}p\int_0^t\left[E\left(\left|\bar{u}^{n,m}_{s\wedge\tau_M^{n,m}\wedge\tau_{\delta_0}^{n,m}} - \bar{u}^n_{s\wedge\tau_M^{n,m}\wedge\tau_{\delta_0}^{n,m}}\right|\right)^{2p}\right]^{\frac{2}{p}} ds. \tag{4.22}$$

Applying Gronwall's inequality to (4.22), one obtains

$$\left[E\left(\sup_{0\leqslant s\leqslant 1\wedge\tau_M^{n,m}\wedge\tau_{\delta_0}^{n,m}}\left|\bar{u}_s^{n,m} - \bar{u}_s^n\right|^{2p}\right)\right]^{\frac{2}{p}}$$

$$\leqslant\left[2\left(\frac{4}{n}\sum_{i=m+1}^{\infty}\lambda_i + 16|\mathrm{b}_m - \mathrm{b}|^2 + \frac{c}{\nu}(\delta_0 M)^{\frac{3}{2}}\right)^2 C_M^2 + \frac{cC_M^2}{n}p\left(\sum_{i=m+1}^{\infty}\lambda_i^2\right)^2\right]$$

$$\times\exp\left(\frac{cC_M^2}{n}p\right). \tag{4.23}$$

Take $p = 2n$ to obtain

$$\lim_{m\to\infty}\limsup_{n\to\infty}\frac{1}{n}\log P\left(\sup_{0\leqslant s\leqslant 1\wedge\tau_M^{n,m}\wedge\tau_{\delta_0}^{n,m}}\left|\bar{u}_s^{n,m} - \bar{u}_s^n\right|^2 > \delta^2\right)$$

$$\leqslant\lim_{m\to\infty}\limsup_{n\to\infty}\log\left[E\left(\sup_{0\leqslant s\leqslant 1\wedge\tau_M^{n,m}\wedge\tau_{\delta_0}^{n,m}}\left|\bar{u}_s^{n,m} - \bar{u}_s^n\right|^{2p}\right)\right]^{\frac{2}{p}} - \log\delta^4$$

$$\leqslant 2\log\left(\frac{\sqrt{2}c}{\nu}(\delta_0 M)^{\frac{3}{2}}C_M\right) + 2cC_M^2 - 4\log\delta. \tag{4.24}$$

Because of Lemma 3.2 and Lemma 3.3, for any $R > 0$, there exists a $M > 0$ such that

$$\lim_{m\to+\infty}\limsup_{n\to+\infty}\frac{1}{n}\log P\left(\left(A^{n,m}\right)^c\right)\leqslant -R,\qquad \lim_{m\to+\infty}\limsup_{n\to+\infty}\frac{1}{n}\log P\left(\left(B^{n,m}\right)^c\right)\leqslant -R. \tag{4.25}$$

By Lemma 5.6 in [16] and Lemma 3.4.5, for any $\delta_0 > 0$, we have

$$\lim_{m\to+\infty}\limsup_{n\to+\infty}\frac{1}{n}\log P\left(\left(C^{n,m}\right)^c\right) = -\infty.$$

Thus for the above choice of M, we have

$$\begin{aligned}
&\lim_{m\to+\infty}\limsup_{n\to+\infty}\frac{1}{n}\log P\Big(\sup_{0\leqslant t\leqslant 1}\big|\bar{u}_t^{n,m}-\bar{u}_t^n\big|^2>\delta^2\Big)\\
&\quad\leqslant\lim_{m\to+\infty}\limsup_{n\to+\infty}\frac{1}{n}\log P\Big(\sup_{0\leqslant t\leqslant 1\wedge\tau_M^{n,m}\wedge\tau_{\delta_0}^{n,m}}\big|\bar{u}_t^{n,m}-\bar{u}_t^n\big|^2>\delta^2\Big)\\
&\qquad\vee\lim_{m\to+\infty}\limsup_{n\to+\infty}\frac{1}{n}\log P\big(\big(A^{n,m}\big)^c\big)\vee\lim_{m\to+\infty}\limsup_{n\to+\infty}\frac{1}{n}\log P\big(\big(B^{n,m}\big)^c\big)\\
&\qquad\vee\lim_{m\to+\infty}\limsup_{n\to+\infty}\frac{1}{n}\log P\big(\big(C^{n,m}\big)^c\big)\\
&\quad\leqslant\left(2\log\left(\frac{\sqrt{2}c}{\nu}(\delta_0 M)^{\frac{3}{2}}C_M\right)+2cC_M^2-4\log\delta\right)\vee -R.
\end{aligned}\tag{4.26}$$

Letting δ_0 go to 0, one obtains

$$\lim_{m\to+\infty}\limsup_{n\to+\infty}\frac{1}{n}\log P\Big(\sup_{0\leqslant t\leqslant 1}\big|\bar{u}_t^{n,m}-\bar{u}_t^n\big|^2>\delta^2\Big)\leqslant -R.\tag{4.27}$$

Since R is arbitrary, (4.11) follows, hence the lemma. $\square$

For $g\in D([0,1];H)$, define $\phi_t^m(g)$ as the solution of the following equation:

$$\phi_t^m(g)=x-\int_0^t A\phi_s^m(g)\,ds-\int_0^t B\big(\phi_s^m(g)\big)\,ds+P_m g(t).$$

Lemma 4.3. *For any $r>0$,*

$$\lim_{m\to\infty}\sup_{\{g:\ I_0(g)\leqslant r\}}\sup_{0\leqslant t\leqslant 1}\big|\phi_t^m(g)-\phi_t(g)\big|=0.$$

Proof. Let $g\in\{g:\ I_0(g)\leqslant r\}$. Let $Z_t^m(g)$ and $Z_t(g)$ be the solutions of the following equations,

$$Z_t^m(g)=-\int_0^t AZ_s^m(g)\,ds+\int_0^t P_m g'(s)\,ds,$$

and

$$Z_t(g)=-\int_0^t AZ_s(g)\,ds+\int_0^t g'(s)\,ds.$$

Set $v_t^m(g):=\phi_t^m(g)-Z_t^m(g)$, $v_t(g):=\phi_t(g)-Z_t(g)$. Then $v_t^m(g)$, $v_t(g)$ satisfy

$$v_t^m(g) = x - \int_0^t A v_s^m(g)\,ds - \int_0^t B\big(v_s^m(g) + Z_s^m(g)\big)\,ds, \tag{4.28}$$

and

$$v_t(g) = x - \int_0^t A v_s(g)\,ds - \int_0^t B\big(v_s(g) + Z_s(g)\big)\,ds. \tag{4.29}$$

As,

$$\frac{1}{2}\frac{d}{dt}\big|Z_t(g)\big|^2 = -\nu\big\|Z_t(g)\big\|^2 + \big(g'(t), Z_t(g)\big),$$

we have,

$$\sup_{0\leqslant t\leqslant 1}\big|Z_t(g)\big|^2 + 2\nu\int_0^1\big\|Z_s(g)\big\|^2\,ds \leqslant \frac{1}{2}\sup_{0\leqslant t\leqslant 1}\big|Z_t(g)\big|^2 + 8\left(\int_0^1\big|g'(s)\big|\,ds\right)^2,$$

so that

$$\sup_{0\leqslant t\leqslant 1}\big|Z_t(g)\big|^2 + 4\nu\int_0^1\big\|Z_s(g)\big\|^2\,ds \leqslant 16\left(\int_0^1\big|g'(s)\big|\,ds\right)^2. \tag{4.30}$$

Note that (4.30) also holds for $Z_t^m(g)$. By Lemma 5.3 in [16], it follows that there exists a M such that

$$\sup_{\{g:\ I_0(g)\leqslant r\}}\int_0^1\big|g'(s)\big|\,ds \leqslant M.$$

Hence,

$$\sup_{\{g:\ I_0(g)\leqslant r\}}\sup_{0\leqslant t\leqslant 1}\big|Z_t(g)\big|^2 \leqslant C_M^1, \qquad \sup_{\{g:\ I_0(g)\leqslant r\}}\int_0^1\big\|Z_t(g)\big\|^2\,dt \leqslant C_M^2, \tag{4.31}$$

where C_M^1, C_M^2 are the constants depending on M. By the chain rule,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\big|v_t(g)\big|^2 &= -\nu\big\|v_t(g)\big\|^2 - b\big(v_t(g) + Z_t(g), v_t(g) + Z_t(g), v_t(g)\big)\\ &\leqslant -\nu\big\|v_t(g)\big\|^2 + \big|b\big(v_t(g), Z_t(g), v_t(g)\big)\big| + \big|b\big(Z_t(g), Z_t(g), v_t(g)\big)\big|\\ &\leqslant -\nu\big\|v_t(g)\big\|^2 + 2\big|v_t(g)\big|\cdot\big\|v_t(g)\big\|\cdot\big\|Z_t(g)\big\| + 2\big|Z_t(g)\big|\cdot\big\|v_t(g)\big\|\cdot\big\|Z_t(g)\big\|.\end{aligned}$$

It follows that

$$\left|v_t(g)\right|^2+\nu\int_0^t\left\|v_s(g)\right\|^2 ds\leqslant\frac{8}{\nu}\int_0^t\left|v_s(g)\right|^2\cdot\left\|Z_s(g)\right\|^2 ds+\frac{8}{\nu}\int_0^t\left|Z_s(g)\right|^2\cdot\left\|Z_s(g)\right\|^2 ds.$$

In view of (4.31), applying Gronwall's inequality, we obtain

$$\sup_{\{g:\ I_0(g)\leqslant r\}}\sup_{0\leqslant t\leqslant 1}\left|v_t(g)\right|^2\leqslant C_M^3,\qquad \sup_{\{g:\ I_0(g)\leqslant r\}}\int_0^1\left\|v_t(g)\right\|^2 dt\leqslant C_M^4,$$

where C_M^3, C_M^4 are the constants depending on M.

It follows from (4.28), (4.29) that,

$$\begin{aligned}&\left|v_t^m(g)-v_t(g)\right|^2+2\nu\int_0^t\left\|v_s^m(g)-v_s(g)\right\|^2 ds\\&\quad\leqslant 2\int_0^t\left|\left\langle B\left(v_s^m(g)+Z_s^m(g)\right)-B\left(v_s(g)+Z_s(g)\right),v_s^m(g)-v_s(g)\right\rangle\right| ds.\end{aligned}$$

By a similar estimate to (4.15), it turns out that

$$\begin{aligned}&\left|v_t^m(g)-v_t(g)\right|^2+2\nu\int_0^t\left\|v_s^m(g)-v_s(g)\right\|^2 ds\\&\quad\leqslant \nu\int_0^t\left\|v_s^m(g)-v_s(g)\right\|^2 ds+\sup_{\{g:\ I_0(g)\leqslant r\}}\sup_{0\leqslant t\leqslant 1}\left|Z_t^m(g)-Z_t(g)\right|C_M\\&\qquad+\nu\int_0^t\left|v_s^m(g)-v_s(g)\right|^2\left(\left\|v_s(g)\right\|^2+\left\|Z_s^m(g)\right\|^2\right) ds,\end{aligned}$$

where C_M is the constant depending on M. By Gronwall's inequality, one obtains

$$\sup_{\{g:\ I_0(g)\leqslant r\}}\sup_{0\leqslant t\leqslant 1}\left|v_t^m(g)-v_t(g)\right|^2\leqslant\left(\sup_{\{g:\ I_0(g)\leqslant r\}}\sup_{0\leqslant t\leqslant 1}\left|Z_t^m(g)-Z_t(g)\right|C_M\right)\cdot C_M. \tag{4.32}$$

By Lemma 5.7 in [16], we know that

$$\lim_{m\to\infty}\sup_{\{g:\ I_0(g)\leqslant r\}}\sup_{0\leqslant t\leqslant 1}\left|Z_t^m(g)-Z_t(g)\right|^2=0.$$

Letting $m\to\infty$ on the both sides of (4.32), we prove the lemma. $\square$

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